

# ALTERNATIVE PROOFS OF A FORMULA FOR BERNOULLI NUMBERS IN TERMS OF STIRLING NUMBERS

BAI-NI GUO AND FENG QI

ABSTRACT. In the paper, the authors provide four alternative proofs of an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind.

## 1. INTRODUCTION

It is well known that Bernoulli numbers  $B_k$  for  $k \geq 0$  may be generated by

$$(1.1) \quad \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi.$$

In combinatorics, Stirling numbers of the second kind  $S(n, k)$  for  $n \geq k \geq 0$  may be computed by

$$(1.2) \quad S(n, k) = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n$$

and may be generated by

$$(1.3) \quad \frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}.$$

In [5, p. 536] and [6, p. 560], the following simple formula for computing Bernoulli numbers  $B_n$  in terms of Stirling numbers of the second kind  $S(n, k)$  was incidentally obtained.

**Theorem 1.1.** *For  $n \in \{0\} \cup \mathbb{N}$ , we have*

$$(1.4) \quad B_n = \sum_{k=0}^n (-1)^k \frac{k!}{k+1} S(n, k).$$

The aim of this paper is to provide four alternative proofs for the explicit formula (1.4).

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## 2. FOUR ALTERNATIVE PROOFS OF THE FORMULA (1.4)

Now we start out to provide four alternative proofs for the explicit formula (1.4).

Considering  $S(0, 0) = 1$ , it is clear that the formula (1.4) is valid for  $n = 0$ . Further considering  $S(n, 0) = 0$  for  $n \geq 1$ , it is sufficient to show

$$(2.1) \quad B_n = \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k), \quad n \in \mathbb{N}.$$

*First proof.* It is listed in [1, p. 230, 5.1.32] that

$$(2.2) \quad \ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} \mathrm{d} u.$$

Taking  $a = 1$  and  $b = 1 + x$  in (2.2) yields

$$(2.3) \quad \frac{\ln(1+x)}{x} = \int_0^\infty \frac{1 - e^{-xu}}{xu} e^{-u} \mathrm{d} u = \int_0^\infty \left( \int_{1/e}^1 t^{xu-1} \mathrm{d} t \right) e^{-u} \mathrm{d} u.$$

Replacing  $x$  by  $e^x - 1$  in (2.3) results in

$$(2.4) \quad \frac{x}{e^x - 1} = \int_0^\infty \left( \int_{1/e}^1 t^{ue^x - u - 1} \mathrm{d} t \right) e^{-u} \mathrm{d} u.$$

In combinatorics, Bell polynomials of the second kind, or say, the partial Bell polynomials,  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  are defined by

$$(2.5) \quad B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i}$$

for  $n \geq k \geq 1$ , see [4, p. 134, Theorem A], and satisfy

$$(2.6) \quad B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

and

$$(2.7) \quad B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) = S(n, k),$$

see [4, p. 135], where  $a$  and  $b$  are any complex numbers. The well-known Faà di Bruno formula may be described in terms of Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  by

$$(2.8) \quad \frac{\mathrm{d}^n}{\mathrm{d} x^n} f \circ g(x) = \sum_{k=1}^n f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)),$$

see [4, p. 139, Theorem C].

Applying in (2.8) the functions  $f(y) = t^y$  and  $g(x) = ue^x - u - 1$  gives

$$(2.9) \quad \frac{\mathrm{d}^n t^{ue^x}}{\mathrm{d} x^n} = \sum_{k=1}^n (\ln t)^k t^{ue^x} B_{n,k}(\overbrace{ue^x, ue^x, \dots, ue^x}^{n-k+1}).$$

Making use of the formulas (2.6) and (2.7) in (2.9) reveals

$$(2.10) \quad \frac{\mathrm{d}^n t^{ue^x}}{\mathrm{d} x^n} = t^{ue^x} \sum_{k=1}^n S(n, k) u^k (\ln t)^k e^{kx}.$$

Differentiating  $n$  times on both sides of (2.4) and considering (2.10) figure out

$$(2.11) \quad \frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right) = \sum_{k=1}^n S(n, k) e^{kx} \int_0^\infty u^k \left( \int_{1/e}^1 (\ln t)^k t^{ue^x - u - 1} dt \right) e^{-u} du.$$

On the other hand, differentiating  $n$  times on both sides of (1.1) gives

$$(2.12) \quad \frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right) = \sum_{k=n}^\infty B_k \frac{x^{k-n}}{(k-n)!}.$$

Equating (2.11) and (2.12) and taking the limit  $x \rightarrow 0$  discover

$$\begin{aligned} B_n &= \sum_{k=1}^n S(n, k) \int_0^\infty u^k \left( \int_{1/e}^1 \frac{(\ln t)^k}{t} dt \right) e^{-u} du \\ &= \sum_{k=1}^n \frac{(-1)^k}{k+1} S(n, k) \int_0^\infty u^k e^{-u} du \\ &= \sum_{k=1}^n \frac{(-1)^k k!}{k+1} S(n, k). \end{aligned}$$

The first proof of Theorem 1.1 is complete.  $\square$

*Second proof.* In the book [2, p. 386] and in the papers [3, p. 615] and [12, p. 885], it was given that

$$(2.13) \quad \frac{\ln b - \ln a}{b - a} = \int_0^1 \frac{1}{(1-t)a + tb} dt,$$

where  $a, b > 0$  and  $a \neq b$ . Replacing  $a$  by 1 and  $b$  by  $e^x$  yields

$$(2.14) \quad \frac{x}{e^x - 1} = \int_0^1 \frac{1}{1 + (e^x - 1)t} dt.$$

Applying the functions  $f(y) = \frac{1}{y}$  and  $y = g(x) = 1 + (e^x - 1)t$  in the formula (2.8) and simplifying by (2.6) and (2.7) give

$$\begin{aligned} \frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right) &= \int_0^1 \frac{d^n}{dx^n} \left[ \frac{1}{1 + (e^x - 1)t} \right] dt \\ &= \int_0^1 \sum_{k=1}^n (-1)^k \frac{k!}{[1 + (e^x - 1)t]^{k+1}} B_{n,k}(\overbrace{te^x, te^x, \dots, te^x}^{n-k+1}) dt \\ &= \sum_{k=1}^n (-1)^k k! \int_0^1 \frac{t^k}{[1 + (e^x - 1)t]^{k+1}} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) dt \\ &\rightarrow \sum_{k=1}^n (-1)^k k! \int_0^1 t^k B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) dt, \quad x \rightarrow 0 \\ &= \sum_{k=1}^n (-1)^k k! S(n, k) \int_0^1 t^k dt \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k). \end{aligned}$$

On the other hand, taking the limit  $x \rightarrow 0$  in (2.12) leads to

$$\frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right) = \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} \rightarrow B_n, \quad x \rightarrow 0.$$

The second proof of Theorem 1.1 is thus complete.  $\square$

*Third proof.* Let  $CT[f(x)]$  be the coefficient of  $x^0$  in  $f(x)$ . Then

$$\begin{aligned} \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k) &= \sum_{k=1}^n (-1)^k CT \left[ \frac{n!}{x^n} \frac{(e^x - 1)^k}{k+1} \right] \\ &= n! CT \left[ \frac{1}{x^n} \sum_{k=1}^{\infty} (-1)^k \frac{(e^x - 1)^k}{k+1} \right] \\ &= n! CT \left[ \frac{1}{x^n} \frac{\ln[1 + (e^x - 1)] - (e^x - 1)}{e^x - 1} \right] \\ &= n! CT \left[ \frac{1}{x^n} \frac{x}{e^x - 1} \right] \\ &= B_n. \end{aligned}$$

Thus, the formula (1.4) follows.  $\square$

*Fourth proof.* It is clear that the equation (1.1) may be rewritten as

$$(2.15) \quad \frac{\ln[1 + (e^x - 1)]}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Differentiating  $n$  times on both sides of (2.15) and taking the limit  $x \rightarrow 0$  reveal

$$\begin{aligned} B_n &= \lim_{x \rightarrow 0} \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left( \frac{\ln[1 + (e^x - 1)]}{e^x - 1} \right) \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[ \frac{\ln(1+u)}{u} \right]^{(k)} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}), \quad u = e^x - 1 \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[ \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{u^{\ell-1}}{\ell} \right]^{(k)} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[ \sum_{\ell=k+1}^{\infty} (-1)^{\ell-1} \frac{(\ell-1)!}{(\ell-k-1)! \ell} u^{\ell-k-1} \right] B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \sum_{k=1}^n \lim_{u \rightarrow 0} \left[ \sum_{\ell=k+1}^{\infty} (-1)^{\ell-1} \frac{(\ell-1)!}{(\ell-k-1)! \ell} u^{\ell-k-1} \right] \lim_{x \rightarrow 0} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k). \end{aligned}$$

The fourth proof of Theorem 1.1 is thus complete.  $\square$

*Remark 2.1.* In [10, p. 1128, Corollary], among other things, it was found that

$$(2.16) \quad B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1}$$

for  $k \in \mathbb{N}$ , where  $A_m$  is defined by

$$\sum_{m=1}^n m^k = \sum_{m=0}^{k+1} A_m n^m.$$

In [6, p. 559] and [9, Theorem 2.1], it was collected and recovered that

$$(2.17) \quad \left( \frac{1}{e^x - 1} \right)^{(k)} = (-1)^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left( \frac{1}{e^x - 1} \right)^m, \quad k \in \{0\} \cup \mathbb{N}.$$

In [9, Theorem 3.1], by the identity (2.17), it was obtained that

$$(2.18) \quad B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1, m+1) S(2k, 2k-m)}{\binom{2k}{m}} - \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k, m) S(2k+1, 2k-m+1)}{\binom{2k}{m-1}}, \quad k \in \mathbb{N}.$$

In [14, Theorem 1.4], among other things, it was presented for  $k \in \mathbb{N}$  that

$$(2.19) \quad B_{2k} = \frac{(-1)^{k-1} k}{2^{2(k-1)} (2^{2k} - 1)} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-i-1} (-1)^{i+\ell} \binom{2k}{\ell} (k-i-\ell)^{2k-1}.$$

Recently, a new formula

$$(2.20) \quad B_n = \sum_{i=0}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i)$$

for  $n \in \mathbb{N}$  was discovered in the preprint [13].

*Remark 2.2.* The identities in (2.17) have been generalized and applied in [8, 15].

*Remark 2.3.* This paper is a slightly revised version of the preprint [7].

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(Guo) SCHOOL OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

*E-mail address:* bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com

(Qi) SCHOOL OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

*E-mail address:* qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

*URL:* <http://qifeng618.wordpress.com>